

Numerical Analysis

Chapter 4 Interpolation and Approximation

4.1 Polynomial Interpolation

Goal Given $n + 1$ data points

$$(x_0, y_0), (x_1, y_1), \dots (x_n, y_n),$$

to find the polynomial of degree less than or equal to n that passes through these points.

Remark There is a unique polynomial of degree less than or equal to n passing through $n + 1$ given points. (Give a proof for $n = 2$.)

Linear Interpolation Given two points (x_0, y_0) and (x_1, y_1) , the linear polynomial passing through the two points is the equation of the line passing through the points. One way to write its formula is

$$P_1(x) = y_0 \frac{x_1 - x}{x_1 - x_0} + y_1 \frac{x - x_0}{x_1 - x_0}.$$

Example For the data points $(2, 3)$ and $(5, 7)$ find $P_1(x)$.

Solution:

$$P_1(x) = 3 \frac{5 - x}{5 - 2} + 7 \frac{x - 2}{5 - 2} = (5 - x) + \frac{5}{3}(x - 2)$$

Example For the data points $(0.82, 2.270500)$ and $(0.83, 2.293319)$, find $P_1(x)$ and evaluate $P_1(0.826)$.

Solution:

$$P_1(x) = 2.270500 \frac{.83 - x}{.83 - .82} + 2.293319 \frac{x - .82}{.83 - .82} = 227.0500 (.83 - x) + 229.3319(x - .82)$$

and hence

$$P_1(.826) = 2.2841914.$$

Remark. If $f(x) = e^x$, then $f(.82) \approx 2.270500$, $f(.83) \approx 2.293319$, and $f(.826) \approx 2.2841638$. Note then that $P_1(x)$ is an approximation of $f(x) = e^x$ for $x \in [.82, .83]$.

In general, if $y_0 = f(x_0)$ and $y_1 = f(x_1)$ for some function f , then $P_1(x)$ is a linear approximation of $f(x)$ for all $x \in [x_0, x_1]$.

Quadratic Interpolation If (x_0, y_0) , (x_1, y_1) , (x_2, y_2) , are given data points, then the quadratic polynomial passing through these points can be expressed as

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

The polynomial $P(x)$ given by the above formula is called **Lagrange's interpolating polynomial** and the functions L_0 , L_1 , L_2 are called **Lagrange's interpolating basis functions**.

Remark Note that $\deg(P_2) \leq 2$ and that

$$L_i(x_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

δ_{ij} is called the **Kronecker delta** function.

Example Construct P_2 from the data points $(0, -1)$, $(1, -1)$, $(2, 7)$.

Solution:

$$P_2(x) = (-1) \frac{(x - 1)(x - 2)}{2} + (-1) \frac{x(x - 2)}{-1} + 7 \frac{x(x - 1)}{2} = \frac{-1}{2}(x - 1)(x - 2) + x(x - 2) + \frac{7}{2}x(x - 1)$$

Example See Example 4.1.4 on page 122 of the text.

Higher-Degree Interpolation Given $n + 1$ data points

$$(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n),$$

the n Lagrange interpolating polynomial is given by

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + \dots + y_n L_n(x)$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \cdots (x_0 - x_n)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_n)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3) \cdots (x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \cdots (x_2 - x_n)}$$

\vdots

$$L_n(x) = \frac{(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1})}$$

Newton's Divided Difference Given distinct points x_0 and x_1 in the domain of a function f , we define

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

This is called the **first-order divided difference** of $f(x)$.

Remark. Note that if f is differentiable on $[x_0, x_1]$, then by Mean Value Theorem, there exists a $c \in (x_0, x_1)$ such that $f[x_0, x_1] = f'(c)$. Furthermore, if x_0 and x_1 are close to each other, then we have

$$f[x_0, x_1] \approx f'(d) \quad \text{with} \quad d = \frac{x_0 + x_1}{2}.$$

Example Consider $f(x) = \cos x$, $x_0 = 0.2$, and $x_1 = 0.3$. Compute $f[x_0, x_1]$.

Solution:

$$f[x_0, x_1] = \frac{\cos(0.3) - \cos(0.2)}{0.3 - 0.2} \approx -0.2473009$$

Note that

$$f'\left(\frac{x_0 + x_1}{2}\right) = -\sin(0.25) \approx -0.247404$$

Definition **Higher order divided differences** are defined recursively using the lower-order ones.

Suppose x_0, x_1, x_2 are distinct point in the domain of f . Then the **second-order divided difference** is given by

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Suppose x_0, x_1, x_2, x_3 are distinct points in the domain of f . Then the **third-order divided difference** is given by

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

In general, if $x_0, x_1, x_2, \dots, x_n$ are distinct points in the domain of f , then the **n th-order divided difference** is given by

$$f[x_0, x_1, x_2, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

Theorem Suppose $x_0, x_1, x_2, \dots, x_n$ are distinct points in $[a, b]$ and suppose f is n times continuously differentiable. Then there exists a point c between the smallest and largest of x_0, x_1, \dots, x_n such that

$$f[x_0, x_1, \dots, x_n] = \frac{f^{(n)}(c)}{n!}.$$

Example Let $f(x) = \cos x$, $x_0 = 0.2$, $x_1 = 0.3$, $x_2 = 0.4$. Compute $f[x_0, x_1, x_2]$.

Solution: From the previous example, we have $f[x_0, x_1] \approx -0.2473009$ and

$$f[x_1, x_2] = \frac{\cos(0.4) - \cos(0.3)}{0.4 - 0.3} \approx -0.3427550$$

Thus

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \approx \frac{-0.3427550 - (-0.2473009)}{0.4 - 0.2} \approx -0.4772705.$$

With $n = 2$ and $c = 0.3$ (a point between 0.2 and 0.3) we have

$$\frac{f''(c)}{2} = -\frac{1}{2} \cos(0.3) \approx -0.4776682$$

which is very close to $f[x_0, x_1, x_2]$ as claimed in the theorem.

Basic Properties of Divided differences

1) $f[x_0, x_1] = f[x_1, x_0]$ and $f[x_0, x_1, x_2] = f[x_1, x_0, x_2] = f[x_1, x_2, x_0] = \dots$. In general if $\{i_0, i_2, \dots, i_n\}$ is a permutation of $\{0, 1, 2, \dots, n\}$, then

$$f[x_0, x_1, \dots, x_n] = f[x_{i_0}, x_{i_1}, \dots, x_{i_n}]$$

2)

$$f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_1 - x_2)} + \frac{f(x_1)}{(x_1 - x_2)(x_2 - x_0)} + \frac{f(x_2)}{(x_2 - x_0)(x_0 - x_1)}$$

3) From the definition we have

$$\lim_{x_1 \rightarrow x_0} f[x_0, x_1] = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0).$$

The we can define

$$f[x_0, x_0] = f'(x_0)$$

In general, if $x_0 = x_1 = x_2 = \dots = x_n$, then

$$f[x_0, x_0, \dots, x_0] = \frac{f^{(n)}(x_0)}{n!}.$$

4) If $x_0 = x_2 \neq x_1$, then

$$f[x_0, x_1, x_0] = f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}$$

Newton's Divided Difference Interpolating Polynomial Or Newton's Form

Define

$$P_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$$

$$P_2(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_1)$$

$$= P_1(x) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$P_3(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$= P_2(x) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

\vdots

$$P_n(x) = P_{n-1}(x) + f[x_0, x_1, \dots, x_{n-1}](x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

The polynomial P_n is called **Newton's divided difference formula for the interpolating polynomial** or **Newton's form for the interpolating polynomial**. Note that $P_n(x_i) = f(x_i)$.

—**Example** Determine the Newton form for the interpolating polynomial for the data set $\{(-1, 5), (0, 1), (1, 1), (2, 5)\}$. Then use this polynomial to approximate y if $x = 1.5$.

Solution

i	x_i	$f[x_i] = f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_0, x_1, x_2, x_3]$
0	-1	5			
1	0	1	-4		
			0		1
2	1	1		5	
				10	
3	2	11			

Therefore

$$\begin{aligned}
 P_3(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\
 &= 5 - 4(x - (-1)) + 2(x - (-1))(x - 0) + 1(x - (-1))(x - 0)(x - 2) \\
 &= 5 - 4(x + 1) + 2x(x + 1) + x(x + 1)(x - 1)
 \end{aligned}$$

And so $P_3(1.5) = 4.375$.

4.2 Error in Polynomial Interpolation

Theorem Let f be a given function on $[a, b]$ and P_n be the polynomial of degree less than or equal to n interpolating the f at the $n + 1$ data points $x_0, x_1, x_2, \dots, x_n$ in $[a, b]$:

$$P_n(x) = f(x_0) \cdot L_0(x) + f(x_1) \cdot L_1(x) + f(x_2) \cdot L_2(x) + \dots + f(x_n) \cdot L_n(x).$$

If f has $n + 1$ continuous derivatives and x_j are distinct, then

$$f(x) - P_n(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(n + 1)!} f^{(n+1)}(c_x)$$

where $a \leq x \leq b$ and c_x is between the maximum and minimum of $x, x_0, x_1, x_2, \dots, x_n$.

Example Let $f(x) = e^x$ on $[0, 1]$ and let $0 \leq x_0 < x_1 \leq 1$. Then by the theorem,

$$f(x) - P_1(x) = \frac{(x - x_0)(x - x_1)}{2} e^{c_x}$$

where $0 \leq x \leq 1$ and c_x is between the maximum and minimum of x, x_0 , and x_1 . If we assume that $x_0 \leq x \leq x_1$, then c_x is between x_0 and x_1 , and we have

$$\frac{(x - x_0)(x_1 - x)}{2} e^{x_0} \leq |f(x) - P_1(x)| \leq \frac{(x - x_0)(x_1 - x)}{2} e^{x_1}$$

Note that if $h = x_1 - x_0$, then

$$\max_{x_0 \leq x \leq x_1} \frac{(x - x_0)(x_1 - x)}{2} = \frac{h^2}{8}$$

and hence

$$|f(x) - P_1(x)| \leq \frac{(x - x_0)(x_1 - x)}{2} e^{x_1} \leq \frac{h^2}{8} e.$$

In particular, if $x_0 = 0.82$, $x_1 = 0.83$ and if $x = 0.826$, then the above reduces to

$$|e^x - P_1(x)| \leq 0.000340$$

Note that the actual error is -0.0000276 .

Example Let $f(x) = e^x$ on $[0, 1]$ and let $0 \leq x_0 < x_1 < x_2 \leq 1$. Then by the theorem,

$$f(x) - P_2(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{6} e^{c_x}$$

where $0 \leq x \leq 1$ and c_x is between the maximum and minimum of x, x_0, x_1 , and x_2 . If we assume that $x_0 \leq x \leq x_2$ and that $h = x_1 - x_0 = x_2 - x_1$, then c_x is between x_0 and x_2 , and we have

$$|f(x) - P_2(x)| \leq \left| \frac{(x - x_0)(x_1 - x)(x - x_2)}{6} e^{x_2} \right| \leq \left| \frac{(x - x_0)(x_1 - x)(x - x_2)}{6} \right| e$$

Note that for $h = x_1 - x_0 = x_2 - x_1$, we have

$$\max_{x_0 \leq x \leq x_2} \left| \frac{(x - x_0)(x_1 - x)}{2} \right| = \frac{h^3}{9\sqrt{3}}.$$

Thus,

$$|f(x) - P_2(x)| \leq \frac{h^3}{9\sqrt{3}} e \approx 0.174h^3.$$

In particular, if $h = 0.01$ then the above reduces to

$$|e^x - P_2(x)| \leq 1.74 \cdot 10^{-7}$$

4.3 Interpolation Using Splines

Remark Consider the data points $(0, 2.5)$, $(1, 0.5)$, $(2, 0.5)$, $(2.5, 1.5)$, $(3, 1.5)$, $(3.5, 1.125)$, $(4, 0)$. The iterating polynomial of Newton and Lagrange are of degree 6. Figure 4.8 on page 148 shows the graph of $P_6(x)$. Such polynomials are not always easy to evaluate and there may be loss of significant digits involved in their calculations.

For these reasons it is desirable to consider **piecewise polynomial interpolation**. This involves finding a continuous function g on $[0, 4]$ and that is a polynomial of 'small' degree in each of the intervals $[0, 1]$, $[1, 2]$, $[2, 2.5]$, $[2.5, 3]$, $[3, 3.5]$, and $[3.5, 4]$. Clearly we need g to interpolate the data set.

Such a function g is called a **piecewise linear interpolation** if each of the polynomials on the subintervals are of degree less than or equal to 1.

We say g is a **piecewise quadratic interpolation** if each of the polynomials on the subintervals are of degree less than or equal to 2.

Example For a **piecewise linear interpolation** of the above data points, see Figure 4.7 on page 147 of your text. Figure 4.9 at the bottom of page 148 shows a **piecewise quadratic interpolation**.

Remark Both the linear and the quadratic interpolating functions are inadequate in that the function g is not differentiable at the **node points**. Thus if smoothness at the node points is required we need the degree of the polynomials to be at least less than or equal to three. As the following theorem states this is all we need.

Theorem If $a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, then there is a unique interpolating function $s(x)$ of the data points (x_1, y_1) , (x_2, y_2) , \cdots , (x_n, y_n) such that

S1 $s(x)$ is a polynomial of degree ≤ 3 on each of the subintervals $[x_{i-1}, x_i]$ for $i = 2, 3, \cdots, n$.

S2 $s(x)$, $s'(x)$, and $s''(x)$ are all continuous on $[a, b]$

S3 $s''(x_1) = s''(x_n) = 0$

The function s satisfying the above theorem is called the **natural cubic spline**

Construction of the natural cubic spline $s(x)$

To simplify notation, we assume that

$$h = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1} = \frac{b - a}{n}.$$

Note that $s''(x)$ is at most linear (why?). Define

$$M_j = s''(x_j).$$

Then, since $s''(x_{j-1}) = M_{j-1}$ and $s''(x_j) = M_j$ are points on the linear function $s''(x)$ and since $s''(x)$ is the equation of the line passing through the points (x_{j-1}, M_{j-1}) and (x_j, M_j) , we can write its equation as

$$s''(x) = \frac{M_{j-1}}{h}(x_j - x) + \frac{M_j}{h}(x - x_{j-1})$$

We now integrate $s''(x)$ twice and use the continuity of s' and s , and the fact that $s(x_i) = y_i$ to obtain

$$s(x) = \frac{M_{j-1}}{6h}(x_j - x)^3 + \frac{M_j}{6h}(x - x_{j-1})^3 + \frac{y_{j-1}}{h}(x_j - x) + \frac{y_j}{h}(x - x_{j-1}) - \frac{hM_{j-1}}{6}(x_j - x) - \frac{hM_j}{6}(x - x_{j-1})$$

For a general formula we can replace h by $x_j - x_{j-1}$. (See text on page 150.)

The M_j are obtained from the following $n - 2$ equation

$$\frac{h}{6}M_{j-1} + \frac{2h}{3}M_j + \frac{h}{6} = \frac{y_{j+1} - y_j}{h} - \frac{y_j - y_{j-1}}{h}$$

and the two conditions form **S3** of the theorem, which in this case translate to

$$M_1 = 0, \quad M_n = 0$$

.

Example For the data points $(1, 1)$, $(2, 1/2)$, $(3, 1/3)$, $(4, 1/4)$, find the natural cubic spline.

Solution: Here $n = 4$ and $h = 1$. The last system of equation is then

$$\frac{1}{6}M_1 + \frac{2}{3}M_2 + \frac{1}{6}M_3 = \frac{1}{3}$$

$$\frac{1}{6}M_2 + \frac{2}{6}M_3 + \frac{1}{6}M_4 = \frac{1}{12}$$

Since $M_1 = M_4 = 0$, the above reduces to

$$\frac{2}{3}M_2 + \frac{1}{6}M_3 = \frac{1}{3}$$

$$\frac{1}{6}M_2 + \frac{2}{6}M_3 = \frac{1}{12}$$

Solving this we get $M_2 = 1/2$ and $M_3 = 0$

Substituting $M_1 = M_3 = M_4 = 0$, $M_2 = 1/2$ and $y_1 = 1, y_2 = 1/2, y_3 = 1/3, y_4 = 1/4$ in the formula for $s(x)$ and simplifying we get

$$s(x) = \begin{cases} \frac{1}{12}x^3 - \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{3}, & 1 \leq x \leq 2 \\ -\frac{1}{12}x^3 - \frac{3}{4}x^2 - \frac{7}{3}x + \frac{17}{6}, & 2 \leq x \leq 3 \\ -\frac{1}{12}x + \frac{7}{12}, & 3 \leq x \leq 4. \end{cases}$$